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MINIMAL SUBMANIFOLDS WITH M-INDEX 2

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For a submanifold M in a Riemannian manifold $\bar{M}$, the *minimal index* (M-index) at a point of M is defined by the dimension of the linear space of all 2nd fundamental forms with vanishing trace. The *geodesic codimension* of M in $\bar{M}$ is defined by the minimum of codimensions of M in totally geodesic submanifolds of $\bar{M}$ containing M.

It is clear that M-index $\leq$ geodesic codimension. In [4, Theorem 1], the author proved that if $\bar{M}$ is of constant curvature, and M is minimal and of M-index 1 at each point, then its geodesic codimension is one. The purpose of the present paper is to investigate an analogous problem for minimal submanifolds with M-index 2. We shall obtain a condition for the geodesic codimension to become 2 (Theorem 1) and some examples (in § 5) of minimal submanifolds with M-index 2 and geodesic codimension 3 in the space forms.

1. Minimal submanifolds with M-index 2

Let $\bar{M}=\bar{M}^{n+\nu}$ be a Riemannian manifold of dimension $n+\nu$ and constant curvature $\bar{c}$, and $M = M^n$ be an n-dimensional submanifold in $\bar{M}$. Let $\bar{\omega}_A$, $\bar{\omega}_{AB} = -\bar{\omega}_{BA}$ $(A, B = 1, 2, \cdots, n+\nu)$ be the basic and connection forms of $\bar{M}$ in the orthonormal frame bundle $F(\bar{M})$ which satisfy the structure equations

$$(1.1)\quad d\bar{\omega}_A = \sum_B \bar{\omega}_{AB} \wedge \bar{\omega}_B\,, \qquad d\bar{\omega}_{AB} = \sum_C \omega_{AC} \wedge \bar{\omega}_{CB} - \bar{c}\omega_A \wedge \bar{\omega}_B\,.$$

Let B be the subbundle of $F(\bar{M})$ over M such that $b = (x, e_1, \cdots, e_n, \cdots, e_{n+\nu}) \in F(\bar{M})$ and $(x, e_1, \cdots, e_n) \in F(M)$, where $F(M)$ is the orthonormal frame bundle of M with the induced Riemannian metric from $\bar{M}$. Then deleting the bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$ in B we have[1]

$$(1.2)\qquad \omega_\alpha = 0\,, \quad \omega_{i\alpha} = \sum_j A_{\alpha ij}\omega_j\,, \quad A_{\alpha ij} = A_{\alpha ji}$$

and

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[1] In the following, $i, j, k, \cdots$ run from 1 to n, and $\alpha, \beta, \gamma, \cdots$ from $n+1$ to $n+\nu$.

$$
\begin{aligned}
d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j , \\
d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} - \bar{c}\omega_i \wedge \omega_j , \\
d\omega_{i\alpha} &= \sum_k \omega_{ik} \wedge \omega_{k\alpha} + \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha} , \\
d\omega_{\alpha\beta} &= - \sum_i \omega_{i\alpha} \wedge \omega_{j\beta} + \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} .
\end{aligned} \tag{1.3}
$$

For any point $x \in M$, let N_x be the normal component to the tangent space $T_xM = M_x$ of $T_x\bar{M} = \bar{M}_x$. Denoting the set of all symmetric real matrices of order n by S_n, for any $b \in B$ we define a linear mapping $\varphi_b \colon N_x \to S_n$ by

$$
\varphi_b(\sum_\alpha v_\alpha e_\alpha) = \sum_\alpha v_\alpha A_\alpha , \quad \text{where} \quad A_\alpha = (A_{\alpha ij}) . \tag{1.4}
$$

Now suppose that M is minimal in $\bar{M}$ and of M-index 2 at each point. Then

$$
\operatorname{trace} A_\alpha = 0 , \qquad \alpha = n+1, \cdots, n+\nu , \tag{1.5}
$$

and N_x is decomposed as $N_x = O_x + \hat{N}_x$, $O_x = \varphi_b{}^{-1}(0)$, $O_x \perp \hat{N}_x$ and $\dim \hat{N}_x = 2$, which does not depend on the choice of b over x and is smooth. Let B_1 be the set of b such that $e_{n+1}, e_{n+2} \in \hat{N}_x$. Then in B_1 we have

$$
\omega_{i,n+3} = \cdots = \omega_{i,n+\nu} = 0 . \tag{1.6}
$$

Lemma 1. *In B_1 for fixed $\beta > n+2$ we have*

$$
\omega_{n+1,\beta} \equiv \omega_{n+2,\beta} \equiv 0 \qquad (\operatorname{mod} \omega_1, \cdots, \omega_n) ,
$$

$$
\omega_{n+1,\beta} = \omega_{n+2,\beta} = 0 \quad \textit{or} \quad \omega_{n+1,\beta} \wedge \omega_{n+2,\beta} \neq 0 .
$$

Proof. Let $\hat{N}$ be the vector bundle over M with fibre $\hat{N}_x$, and take a smooth local cross section $(x, \hat{e}_{n+1}, \hat{e}_{n+2})$ of the orthonormal frame bundle of $\hat{N}$. Then for b we can put

$$
e_{n+1} = \hat{e}_{n+1} \cos\theta_1 + \hat{e}_{n+2} \sin\theta_1 , \quad e_{n+2} = \hat{e}_{n+1} \cos\theta_2 + \hat{e}_{n+2} \sin\theta_2 ,
$$

and we have

$$
\omega_{n+1,\beta} = \hat{\omega}_{n+1,\beta} \cos\theta_1 + \hat{\omega}_{n+2,\beta} \sin\theta_1 , \quad \omega_{n+2,\beta} = \hat{\omega}_{n+1,\beta} \cos\theta_2 + \hat{\omega}_{n+2,\beta} \sin\theta_2 ,
$$

where $\hat{\omega}_{n+1,\beta} = \langle \bar{D}\hat{e}_{n+1}, e_\beta \rangle$, $\hat{\omega}_{n+2,\beta} = \langle \bar{D}e_{n+2}, e_\beta \rangle$, and $\bar{D}$ denotes the covariant differential operator in $\bar{M}$. Thus $\omega_{n+1,\beta} \equiv \omega_{n+2,\beta} \equiv 0 \ (\operatorname{mod} \omega_1, \cdots, \omega_n)$. Next, from $\omega_{i\beta} = 0$ and (1.3) it follows that

$$
\omega_{i,n+1} \wedge \omega_{n+1,\beta} + \omega_{i,n+2} \wedge \omega_{n+2,\beta} = 0 . \tag{1.7}
$$

By assuming $\omega_{n+2,\beta} = \rho\omega_{n+1,\beta}$ at x, (1.7) implies $(\omega_{i,n+1} + \rho\omega_{i,n+2}) \wedge \omega_{n+1,\beta} = 0$.

Since A_{n+1} and A_{n+2} are linearly independent in S_n, $A_{n+1}+\rho A_{n+2}\neq 0$, from which follows rank $(A_{n+1}+\rho A_{n+2})>1$ with trace $(A_{n+1}+\rho A_{n+2})=0$. Hence $\omega_{n+1,\beta}=\omega_{n+2,\beta}=0$. q.e.d.

Now for any $v\in\hat{N}$, we define a linear mapping $\psi_v\colon M_x\to O_x$ by

$$\psi_v(X)=\sum_{\beta>n+2}\langle v, e_{n+1}\omega_{n+1,\beta}(X)+e_{n+2}\omega_{n+2,\beta}(X)\rangle e_\beta\ , \tag{1.8}$$

where $b\in B_1$, $X\in M_x$. ψ_v is well defined by Lemma 1.

The space of relative nullity of M in $\overline{M}$ at x is the set of $X\in M_x$ such that $\omega_{i\alpha}(X)=0$, $i=1,2,\cdots,n$; $\alpha=n+1,\cdots,n+\nu$, which, in general, is denoted by $\mathfrak{l}_x$. Put

$$M_x=\mathfrak{m}_x+\mathfrak{l}_x\ ,\qquad \mathfrak{m}_x\perp\mathfrak{l}_x\ . \tag{1.9}$$

Lemma 2. *If $\omega_{n+1,\beta}\wedge\omega_{n+2,\beta}\neq 0$ for a fixed $\beta>n+2$ in B_1 at $x\in M$, we can choose frames $b\in B_1$ such that $e_1,e_2\in\mathfrak{m}_x$, $e_3,\cdots,e_n\in\mathfrak{l}_x$ and*

$$\begin{aligned}&\omega_{1,n+1}=\lambda\omega_1\ ,\quad \omega_{2,n+1}=-\lambda\omega_2\ ,\quad \omega_{3,n+1}=\cdots=\omega_{n,n+1}=0\ ,\\ &\omega_{1,n+2}=\mu\omega_2\ ,\quad \omega_{2,n+2}=\mu\omega_1\ ,\quad \omega_{3,n+2}=\cdots=\omega_{n,n+2}=0\ ,\\ &\omega_{n+1,\beta}\equiv\omega_{n+2,\beta}\equiv 0 \pmod{\omega_1,\omega_2}\ ,\quad \lambda\neq 0\ ,\quad \mu\neq 0\ .\end{aligned} \tag{1.10}$$

Proof. From (1.7), we have

$$\omega_{i,n+1}\wedge\omega_{n+1,\beta}\wedge\omega_{n+2,\beta}=\omega_{i,n+2}\wedge\omega_{n+1,\beta}\wedge\omega_{n+2,\beta}=0\ .$$

By the assumption and Lemma 1, we can choose frames $(x,e_1,\cdots,e_n)$ such that $\omega_{n+1,\beta}\wedge\omega_{n+2,\beta}=f\omega_1\wedge\omega_2$, $f\neq 0$. Then the above equations imply $\omega_{i,n+1}\equiv\omega_{i,n+2}\equiv 0 \pmod{\omega_1,\omega_2}$, and therefore we can choose $b\in B_1$ such that[2] $\langle A_{n+1},A_{n+2}\rangle=0$ and

$$\omega_{1,n+1}=\lambda\omega_1\ ,\quad \omega_{2,n+1}=-\lambda\omega_2\ ,\quad \omega_{r,n+1}=\omega_{r,n+2}=0\ ,\quad 2<r\leq n\ .$$

Then putting $\omega_{1,n+2}=b_1\omega_1+\mu\omega_2$, $\omega_{2,n+2}=\mu\omega_1+b_2\omega_2$, we have $n\langle A_{n+1},A_{n+2}\rangle=\lambda(b_1-b_2)=0$, so that $b_1=b_2=0$. Thus we obtain (1.10). It is clear that e_1, $e_2\in\mathfrak{m}_x$, and $e_3,\cdots,e_n\in\mathfrak{l}_x$.

Theorem 1. *If M^n is minimal and of M-index 2 in a Riemannian manifold $\overline{M}^{n+\nu}$ of constant curvature $\bar{c}$ at each point, then ψ_v, $v\in\hat{N}_x$, $v\neq 0$, has a common image $\psi_v(M_x)$ whose dimension is at most 2. If the rank of ψ_v is constantly zero for $v\in\hat{N}_x$, then the geodesic codimension of M^n is 2, and M^n is also minimal and of M-index 2 in the geodesic submanifold $\overline{M}^{n+2}$ in $\overline{M}^{n+\nu}$ which contains M^n. If the rank of ψ_v is not zero, then*

[2] In S_n, we define the inner product of any A and B by $\langle A,B\rangle=\operatorname{trace}AB/n$, so that S_n is a Euclidean space.

$$\text{(i)} \quad \dim \mathfrak{l}_x = n - 2 , \qquad \text{(ii)} \quad \psi_v(\mathfrak{l}_x) = 0 .$$

Proof. If ψ_v is trivial for any v, then $\omega_{n+1,\beta} = \omega_{n+2,\beta} = 0$, $\beta > n + 2$, in B_1. On the other hand, the system of Pfaffian equations:

$$(1.11) \qquad \bar{\omega}_\beta = 0 , \quad \bar{\omega}_{i\beta} = 0 , \quad \bar{\omega}_{n+1,\beta} = 0 , \quad \bar{\omega}_{n+2,\beta} = 0 , \quad i = 1, \cdots, n; \ \beta = n + 3, \cdots, n + \nu$$

in $F(\bar{M}^{n+\nu})$ is completely integrable and the image of any maximal integral submanifold under the projection $F(\bar{M}^{n+\nu}) \to \bar{M}^{n+\nu}$ is totally geodesic. Therefore M^n is contained in an $(n + 2)$-dimensional totally geodesic submanifold $\bar{M}^{n+2}$ of $\bar{M}^{n+\nu}$. It is clear that M^n is minimal and of M-index 2 in $\bar{M}^{n+2}$.

Now suppose that ψ_v, $v \in \hat{N}_x$, is not trivial. By (1.8) and Lemma 1, there exists $\beta > n + 2$ such that $\omega_{n+1,\beta} \wedge \omega_{n+2,\beta} \neq 0$. Choosing a frame $b \in B_1$, which satisfies (1.10), and substituting (1.7) we get, for any $\gamma > n + 2$,

$$\lambda\omega_1 \wedge \omega_{n+1,\gamma} + \mu\omega_2 \wedge \omega_{n+2,\gamma} = 0 , \qquad -\lambda\omega_2 \wedge \omega_{n+1,\gamma} + \mu\omega_1 \wedge \omega_{n+2,\gamma} = 0 .$$

Hence we can put

$$(1.12) \qquad \lambda\omega_{n+1,\gamma} = f_\gamma\omega_1 + g_\gamma\omega_2 , \qquad \mu\omega_{n+2,\gamma} = g_\gamma\omega_1 - f_\gamma\omega_2 .$$

By putting $F = \sum_{\gamma > n+2} f_\gamma e_\gamma$, $G = \sum_{\gamma > n+2} g_\gamma e_\gamma$, (1.8) can be written as

$$(1.13) \qquad \begin{aligned} \psi_v(X) &= \left\{\frac{1}{\lambda}\langle v, e_{n+1}\rangle\omega_1(X) - \frac{1}{\mu}\langle v, e_{n+2}\rangle\omega_2(X)\right\} F \\ &\quad + \left\{\frac{1}{\lambda}\langle v, e_{n+1}\rangle\omega_2(X) + \frac{1}{\mu}\langle v, e_{n+2}\rangle\omega_1(X)\right\} G . \end{aligned}$$

Since $\omega_{n+1,\beta} \wedge \omega_{n+2,\beta} \neq 0$, we have $f_\beta{}^2 + g_\beta{}^2 \neq 0$, so that $F \neq 0$ or $G \neq 0$. Since

$$\det \begin{pmatrix} \langle v, e_{n+1}\rangle/\lambda & -\langle v, e_{n+2}\rangle/\mu \\ \langle v, e_{n+2}\rangle/\mu & \langle v, e_{n+1}\rangle/\lambda \end{pmatrix} = \frac{1}{\lambda^2}\langle v, e_{n+1}\rangle^2 + \frac{1}{\mu^2}\langle v, e_{n+2}\rangle^2 > 0$$

for $v \neq 0$, the image $\psi_v(M_x)$ is the linear space spanned by F and G, which does not depend on $v \in \hat{N}_x$, $v \neq 0$. Hence (i) and (ii) are clear by Lemma 2.

Remark. In Theorem 1, the set of $x \in M$ such that ψ_v is not trivial is open. For such points x, by means of (1.12) the frame $b = (x, e_1, \cdots, e_{n+\nu})$ satisfying (1.10) does not depend on the choice of β such that $\omega_{n+1,\beta} \wedge \omega_{n+2,\beta} \neq 0$. In the above open set of M, F and G give normal vector fields, and the set of such frames is denoted by B_2.

2. Minimal submanifolds with M-index 2 and geodesic codimension >2

Using the notations in § 1, we have

Lemma 3. *Suppose the rank of $\psi_v > 0$ for every $v \neq 0$. Then the $(n-2)$-dimensional distribution $\mathfrak{l} = \{\mathfrak{l}_x, x \in M^n\}$ is completely integrable and its integral submanifolds are totally geodesic in $\overline{M}^{n+\nu}$.*

Proof. From $\omega_{r,n+1} = \omega_{r,n+2} = 0$ $(2 < r \leq n)$ it follows that

$$\omega_{r1} \wedge \omega_{1,n+1} + \omega_{r2} \wedge \omega_{2,n+1} = \omega_{r1} \wedge \omega_{1,n+2} + \omega_{r2} \wedge \omega_{2,n+2} = 0$$

in B_2, and from (1.10) that $\omega_{r1} \wedge \omega_1 - \omega_{r2} \wedge \omega_2 = \omega_{r1} \wedge \omega_2 + \omega_{r2} \wedge \omega_1 = 0$. Thus we can put

$$\omega_{1r} = p_r\omega_1 - q_r\omega_2 \,, \qquad \omega_{2r} = q_r\omega_1 + p_r\omega_1 \,, \tag{2.1}$$

or $\omega_{1r} + i\omega_{2r} = (p_r + iq_r)(\omega_1 + i\omega_2)$. Making use of these relations we can easily see that $d\omega_1 = d\omega_2 = 0 \pmod{\omega_1, \omega_2}$. Hence the Pfaffian equations $\omega_1 = \omega_2 = 0$ are completely integrable, and, equivalently, so is the distribution $\mathfrak{l}$.

Let L^{n-2} be a maximal integral submanifold of $\mathfrak{l}$, along which we have $\omega_1 = \omega_2 = \omega_{n+1} = \cdots = \omega_{n+\nu} = 0$ and $\omega_{1r} = \omega_{2r} = \omega_{r,n+1} = \cdots = \omega_{r,n+\nu} = 0$ by (2.1), (1.10) and (1.6) in B_2. These show that L^{n-2} is totally geodesic in $\overline{M}^{n+\nu}$. q.e.d.

In the proof of Lemma 3, we have two special tangent vector fields defined by

$$P = \sum_{r=3}^{n} p_r e_r \,, \qquad Q = \sum_{r=3}^{n} q_r e_r \,, \tag{2.2}$$

which we call the *principal* and *subprincipal asymptotic vector fields*, respectively.

Lemma 4. *Under the condition of Lemma 3, the 2-dimensional distribution $\mathfrak{w} = \{\mathfrak{w}_x, x \in M^n\}$ is completely integrable if and only if the vector field Q vanishes. When $Q = 0$, the integral submanifolds of $\mathfrak{w}$ are totally umbilic in M^n.*

Proof. $\mathfrak{w}_x$ is given by the Pfaffian equations $\omega_3 = \omega_4 = \cdots = \omega_n = 0$ at each point $x \in M^n$. By (2.1), in B_2 we have $d\omega_r \equiv -2q_r\omega_1 \wedge \omega_2 \pmod{\omega_3, \cdots, \omega_n}$, which shows that the distribution $\mathfrak{w}$ is completely integrable if and only if $Q = 0$.

When $Q = 0$, (2.1) becomes

$$\omega_{1r} = p_r\omega_1 \,, \quad \omega_{2r} = p_r\omega_2 \,, \quad r = 3, \cdots, n \,, \tag{2.3}$$

which shows that any integral submanifold of the distribution $\mathfrak{w}$ is totally umbilic in M^n. q.e.d.

We will explain the integrability of $\mathfrak{w}$ without using the field Q.

Lemma 5. *The distribution $\mathfrak{w}$ is completely integrable if and only if the* *following condition is satisfied: For any tangent vector fields $X \subset \mathfrak{w}$, and $Y \subset \mathfrak{l}$, we have $(\nabla_X Y)_{\mathfrak{w}} \parallel X$, where ∇_X denotes the covariant derivative in M^n with respect to X and $(\nabla_X Y)_{\mathfrak{w}}$ the $\mathfrak{w}$-component of the field $\nabla_X Y$.*

Proof. Putting $X = \sum_{a=1}^{2} X^a e_a$, $Y = \sum_{r=3}^{n} Y^r e_r$ and considering e_r as local fields, we have

$$\nabla_X Y = \sum_a X^a \sum_r \Big\{(\nabla_{e_a} Y^r)e_r + Y^r(\omega_{r1}(e_a)e_1 + \omega_{r2}(e_a)e_2) + \sum_{t>2} \omega_{rt}(e_a)e_t\Big\} .$$

Thus by (2.1),

$$(\nabla_X Y)_{\mathfrak{w}} = -(X^1\langle P, Y\rangle - X^2\langle Q, Y\rangle)e_1 - (X^1\langle Q, Y\rangle + X^2\langle P, Y\rangle)e_2 ,$$

that is,

$$(\nabla_X Y)_{\mathfrak{w}} = -\langle P, Y\rangle X - \langle Q, Y\rangle \operatorname{Rot}_{\pi/2} X , \tag{2.4}$$

where $\operatorname{Rot}_{\pi/2}$ denotes the rotation on $\mathfrak{w}_x$ by the angle $\pi/2$ in the direction from e_1 to e_2. Hence $Q = 0$ is equivalent to the statement of this lemma.

Lemma 6. *Suppose the rank of $\psi_v > 0$ for every $v \neq 0$. Then in B_2,*

$$\{(d\lambda - \lambda\langle P, dx\rangle) - i(2\lambda\omega_{12} - \mu\hat{\omega} + \lambda\langle Q, dx\rangle)\} \wedge (\omega_1 + i\omega_2) = 0 , \tag{2.5}$$

$$\{(d\mu - \mu\langle P, dx\rangle) - i(2\mu\omega_{12} - \lambda\hat{\omega} + \mu\langle Q, dx\rangle)\} \wedge (\omega_1 + i\omega_2) = 0 , \tag{2.6}$$

$$\{d\sigma + i(1 - \sigma^2)\hat{\omega}\} \wedge (\omega_1 + i\omega_2) = 0 , \tag{2.7}$$

$$d\omega_{12} = -\{\|P\|^2 + \|Q\|^2 + \bar{c} - \lambda^2 - \mu^2\}\omega_1 \wedge \omega_2 , \tag{2.8}$$

$$d\hat{\omega} = -\frac{1}{\lambda\mu}\{2\lambda^2\mu^2 - \|F\|^2 - \|G\|^2\}\omega_1 \wedge \omega_2 , \tag{2.9}$$

where $\langle P, dx\rangle = \sum_{r=3}^{n} p_r\omega_r$, $\langle Q, dx\rangle = \sum_{r=3}^{n} q_r\omega_r$, $\hat{\omega} = \omega_{n+1,n+2}$ *and* $\sigma = \mu/\lambda$.

Proof. From (1.10), (1.12) and (2.1) we get

$$d\omega_{1,n+1} = -\lambda\omega_{12} \wedge \omega_2 + \mu\hat{\omega} \wedge \omega_2 = d\lambda \wedge \omega_1 + \lambda \sum_{j=1}^{n} \omega_{1j} \wedge \omega_j ,$$

$$d\omega_{2,n+1} = -\lambda\omega_{12} \wedge \omega_1 + \mu\hat{\omega} \wedge \omega_1 = -d\lambda \wedge \omega_2 - \lambda \sum_{j=1}^{n} \omega_{2j} \wedge \omega_j ,$$

and therefore

$$(d\lambda - \lambda \sum_r p_r\omega_r) \wedge \omega_1 + (2\lambda\omega_{12} - \mu\hat{\omega} + \lambda \sum_r q_r\omega_r) \wedge \omega_2 = 0 ,$$
$$(d\lambda - \lambda \sum_r p_r\omega_r) \wedge \omega_2 - (2\lambda\omega_{12} - \mu\hat{\omega} + \lambda \sum_r q_r\omega_r) \wedge \omega_1 = 0 .$$

which can be written as (2.5). Analogously we can get (2.6) from $d\omega_{1,n+2}$ and $d\omega_{2,n+2}$. From (2.5) and (2.6) it is easily seen that

$$\{(\lambda d\mu - \mu d\lambda) + i(\lambda^2 - \mu^2)\} \wedge (\omega_1 + i\omega_2) = 0 ,$$

which is equivalent to (2.7). We have also

$$d\omega_{12} = \sum_r \omega_{1r} \wedge \omega_{r2} + \omega_{1,n+1} \wedge \omega_{n+1,2} + \omega_{1,n+2} \wedge \omega_{n+2,2} - \bar{c}\omega_1 \wedge \omega_2$$
$$= - \left\{\sum_r (p_r^2 + q_r^2) + \bar{c} - \lambda^2 - \mu^2\right\} \omega_1 \wedge \omega_2 ,$$
$$d\hat{\omega} = \sum_{a=1}^{2} \omega_{n+1,a} \wedge \omega_{a,n+2} + \sum_{\beta>n+2} \omega_{n+1,\beta} \wedge \omega_{\beta,n+2}$$
$$= - \frac{1}{\lambda\mu} \left\{2\lambda^2\mu^2 - \sum_\beta (f_\beta^2 + g_\beta^2)\right\} \omega_1 \wedge \omega_2 ,$$

which can be written as (2.8) and (2.9), respectively. q.e.d.

A curve in a Riemannian manifold of constant curvature is said to be *even* if its geodesic codimension ≤ 1.

Theorem 2. *Under the conditions of Theorem* 1 *with non-trivial* ψ_v *for any* $v \in \hat{N}$, $v \neq 0$, *the following statements hold.*

1) *The set of all asymptotic tangent vectors of* M^n *in* $\bar{M}^{n+\nu}$ *constitute a completely integrable* $(n-2)$*-dimensional distribution* $\mathfrak{l}$ *and its integral submanifolds are totally geodesic in* $\bar{M}^{n+\nu}$.

2) *The* 2*-dimensional distribution* $\mathfrak{w}$ *orthogonally complement to* $\mathfrak{l}$ *is completely integrable if and only if the subprincipal asymptotic vector field* Q *of* M^n *vanishes, and then its integral surfaces are totally umblic in* M^n.

3) *The principal and subprincipal asymptotic vector fields* P *and* Q *of* M^n *are involutive.*

4) *When* $P \neq 0$, *the integral curves of* P *are even in* $\bar{M}^{n+\nu}$, *and they are geodesic of* $\bar{M}^{n+\nu}$ *if and only if* $\langle P, Q\rangle = 0$ *or* $P \| Q$.

Proof. 1) and 2) are evident from Lemmas 3 and 4. By (2.1) and (1.3) we obtain

$$d(\omega_{1r} + i\omega_{2r})$$
$$= \sum_j (\omega_{1j} \wedge \omega_{jr} + i\omega_{2j} \wedge \omega_{jr}) - \bar{c}(\omega_1 + i\omega_2) \wedge \omega_r$$
$$= (dp_r + idq_r) \wedge (\omega_1 + i\omega_2) + (p_r + iq_r) \sum_j (\omega_{1j} \wedge \omega_j + i\omega_{2j} \wedge \omega_j) ,$$

and therefore

$$\left\{dp_r + idq_r + \sum_t (p_t + iq_r)\omega_{tr} - (p_r + iq_r) \sum_t (p_t + iq_t)\omega_t - \bar{c}\omega_r\right\} \wedge (\omega_1 + i\omega_2) = 0 . \tag{2.10}$$

from which it follows that for any tangent vector field $X \subset \mathfrak{l}$,

$$\bar{\nabla}_X P = \nabla_X P = \langle P, X\rangle P - \langle Q, X\rangle Q + \bar{c}X , \tag{2.11}$$

$$\bar{\nabla}_X Q = \nabla_X Q = \langle Q, X\rangle P + \langle P, X\rangle Q , \tag{2.12}$$

where $\bar{\nabla}_X$ denotes the covariant derivative in $\bar{M}^{n+\nu}$ with respect to X. In particular, we get $\nabla_Q P = \langle P, Q\rangle P - \|Q\|^2 Q + \bar{c}Q$, $\nabla_P Q = \langle P, Q\rangle P + \|P\|^2 Q$, and therefore $[P, Q] = \nabla_P Q - \nabla_Q P = \{\|P\|^2 + \|Q\|^2 - \bar{c}\}Q$, which shows that P and Q are involutive.

For part 4) of the theorem we notice the following equations derived from (2.11) and (2.12):

$$\bar{\nabla}_P P = (\|P\|^2 + \bar{c})P - \langle P, Q\rangle Q , \qquad \bar{\nabla}_Q Q = \|Q\|^2 P + \langle P, Q\rangle Q ,$$

which clearly show that if $P \wedge Q \neq 0$, then the integral surfaces of the distribution spanned by P and Q are totally geodesic in $\bar{M}^{n+\nu}$. Hence, when $P \neq 0$, the integral curves of P are even, and they are geodesics in $\bar{M}^{n+\nu}$ if and only if $\langle P, Q\rangle Q \,\|\, P$, that is, if and only if $\langle P, Q\rangle = 0$ or $Q \,\|\, P$.

3. Minimal submanifolds with M-index 2 and vanishing subprincipal asymptotic vector field Q

In this section, we shall consider M^n in $\bar{M}^{n+\nu}$ as in Theorem 2 under the additional conditions $P \neq 0$ and $Q = 0$, and suppose $n \geq 3$. Denote the integral surface of $\mathfrak{w}$ and the integral curve of P through x by $W^2(x)$ and $\Gamma^1(x)$ respectively.

Lemma 7. *The integral curves Γ^1 of P are the orthogonal trajectories of a family of hypersurfaces of M^n containing the integral surfaces W^2 of $\mathfrak{w}$.*

Proof. Since $Q \equiv 0$, (2.10) is reduced to

$$dp_r + \sum_{t>2} p_t\omega_{tr} - p_r \sum_{t>2} p_t\omega_t - \bar{c}\omega_r = 0 . \tag{3.1}$$

Since $P \neq 0$, we use only such frames b of B_2 that

$$P = pe_3 , \qquad p > 0 , \tag{3.2}$$

and denote the submanifold of these frames by B_3, in which

$$\omega_{a3} = p\omega_a , \quad \omega_{at} = 0 , \quad a = 1, 2; 3 < t \leq n , \tag{3.3}$$

and (3.1) becomes

(3.4) $$dp = (p^2 + \bar{c})\omega_3 ,$$

(3.5) $$p\omega_{3r} = \bar{c}\omega_r , \qquad 3 < r \leq n .$$

By means of (3.3) and (3.5) we obtain $d\omega_3 = 0$ in B_3, so that there exists a local function v such that

(3.6) $$\omega_3 = dv .$$

(3.2) and (3.6) show that the family of level hypersurfaces of v is the required one.

Remark. By denoting the level hypersurface $v = c$ by $V^{n-1}(c)$, the function v may be considered as the arclength of the geodesics Γ^1 measured from $V^{n-1}(0)$. Integrating (3.4), we easily have

Lemma 8. *The norm p of the principal asymptotic vector field P is a function of v as follows:*

(3.7₁) $$p = (\bar{c})^{-1/2} \tan (v + a)\sqrt{\bar{c}} , \quad 0 < v + a < \pi/(2\sqrt{\bar{c}}), \quad (\bar{c} > 0) .$$

(3.7₂) $$p = 1/(a - v) , \quad v < a , \quad (\bar{c} = 0) .$$

(3.7₃) $$p = \begin{cases} \sqrt{-\bar{c}} \tanh (a - v)\sqrt{-\bar{c}} , & (0 < p < \sqrt{-\bar{c}}) , \\ \sqrt{-\bar{c}} \coth (a - v)\sqrt{-\bar{c}} , & (\sqrt{-\bar{c}} < p) , \end{cases} \quad v < a ,(\bar{c} < 0) .$$

Here a is a constant on M^n.

Lemma 9. *Let X be a Jacobi field along Γ^1 determined by a family of integral geodesics of P. If $X(0) \in \mathfrak{w}$, then $\|X\| \to 0$ and $p \to +\infty$ when $v + a \to \pi/(2\sqrt{\bar{c}})$ for $\bar{c} > 0$ and $v \to a$ for $\bar{c} = 0$, or $\bar{c} < 0$ and $\sqrt{-\bar{c}} < p$.*

Proof. Let $x = x(v, \varepsilon)$ be a family of integral geodesics of P such that $x(v, \varepsilon) \in V^{n-1}(v)$. Putting $X = \partial x/\partial \varepsilon$, we obtain $X^2 = \sum_{j \neq 3} \omega_j(X)\omega_j(X)$ and $\partial \|X\|^2/\partial v = 2 \sum_{j \neq 3} \omega_j(X)\partial\omega_j(X)/\partial v$. On the other hand, we have

$$\begin{aligned} \partial\omega_j(X)/\partial v &= e_3(\omega_j(X)) = X(\omega_j(e_3)) - d\omega_j(X, e_3) - \omega_j([X, e_3]) \\ &= - \sum_k \omega_{jk} \wedge \omega_k(X, e_3) , \end{aligned}$$

since $[\partial/\partial v, \partial/\partial\varepsilon] = 0$ and so $\omega_j([X, e_3]) = 0$. Thus

$$\partial \|X\|^2/\partial v = -2 \sum_a \omega_j(X)\omega_{j3}(X) = -2 \sum_a \omega_a(X)\omega_{a3}(X) + 2 \sum_{r>3} \omega_r(X)\omega_{3r}(X) .$$

Using (3.3) and (3.5), we have

(3.8) $$\partial \|X\|^2/\partial v = -2p \|X_{\mathfrak{w}}\|^2 + 2(\bar{c}/p) \|X_{\mathfrak{l}}\|^2 ,$$

where $X_{\mathfrak{w}}$ and $X_{\mathfrak{l}}$ are the $\mathfrak{w}$ and $\mathfrak{l}$ components of X.

On the other hand, in B_3 we have $d\omega_r = (\bar{c}/p)\omega_3 \wedge \omega_r + \sum_{t>3} \omega_{rt} \wedge \omega_t$, so that the Pfaffian equations $\omega_4 = \cdots = \omega_n = 0$ are completely integrable. Thus, if $X \in \mathfrak{w}$ for a value of v, then so is for any v. For such X from (3.8) it follows that $\partial \|X\|^2/\partial v = -2p\|X\|^2$. Integrating this and using Lemma 8, we have

$$(3.9) \qquad \|X(v)\|/\|X(0)\| = \exp\left(-\int_0^v p dv\right) = \begin{cases} \cos(v+a)\sqrt{\bar{c}}/\cos a\sqrt{\bar{c}} & (\bar{c} > 0)\,, \\ (a-v)/a \; (\bar{c} = 0)\,, \\ \sinh(a-v)\sqrt{-\bar{c}}/\sinh a\sqrt{-\bar{c}} & (\bar{c} < 0 \text{ and } -\bar{c} < p)\,, \\ \cosh(a-v)\sqrt{-\bar{c}}/\cosh a\sqrt{-\bar{c}} & (\bar{c} < 0 \text{ and } 0 < p < -\bar{c})\,, \end{cases}$$

which implies this lemma.

Lemma 10. *Let X be a Jacobi field along Γ^1 as in Lemma 9. If $X(0) \in \mathfrak{l}$, $\langle X(0), P\rangle = 0$, then*

i) *$\|X\| \to 0$ and $p \to 0$, when $v + a \to 0$ for $\bar{c} > 0$,*

ii) *$\|X(v)\| = \|X(0)\|$ for $\bar{c} = 0$,*

iii) *$\|X\| \to 0$ and $p \to 0$, or $\|X\| \to \|X(0)\|/\cos a\sqrt{-\bar{c}}$ and $p \to \infty$ when $v \to a$ for $\bar{c} < 0$.*

Proof. By Lemmas 3 and 7, we have $X \subset \mathfrak{l}$ and $\langle X, P\rangle = 0$ for any v. Thus (3.8) implies $\partial\|X\|^2/\partial v = 2(\bar{c}/p)\|X\|^2$, from which it follows that

$$(3.10) \qquad \|X(v)\|/\|X(0)\| = \exp\left(\bar{c}\int_0^v (1/p)dv\right) = \begin{cases} \sin(v+a)\sqrt{\bar{c}}/\sin a\sqrt{\bar{c}} & (\bar{c} > 0)\,, \\ 1 \quad (\bar{c} = 0)\,, \\ \cosh(a-v)\sqrt{-\bar{c}}/\cosh a\sqrt{-\bar{c}} & (\bar{c} < 0 \text{ and } \sqrt{-\bar{c}} < p)\,, \\ \sinh(a-v)\sqrt{-\bar{c}}/\sinh a\sqrt{-\bar{c}} & (\bar{c} < 0 \text{ and } 0 < p < \sqrt{-\bar{c}})\,. \end{cases}$$

These relations and Lemma 8 imply i), ii) and iii). q.e.d.

By means of Lemmas 7, 9 and Theorem 2, we obtain

Theorem 3. *Let M^n $(n \geq 3)$ be a maximal minimal submanifold*[3] *in an $(n+\nu)$-dimensional space form $\bar{M}^{n+\nu}$ which is of M-index 2 at each point, whose associate mapping ψ_v is nontrivial for any $v \in \hat{N}, v \neq 0$, and subprincipal asymptotic vector field vanishes identically. Then M^n is a locus of $(n-2)$-dimensional totally geodesic subspaces in $L^{n-2}(y)$ in $\bar{M}^{n+\nu}$ through points y of*

[3] "maximal" means here that M^n is not contained in a larger submanifold with the same properties.

a surface W^2 lying in a Riemannian hypersphere in $\bar{M}^{n+\nu}$ with center z_0 such that

i) *$L^{n-2}(y)$ contains the geodesic from z_0 to y,*

ii) *the $(n-3)$-dimensional tangent spaces to the intersection of $L^{n-2}(y)$ and the hypersphere at y are parallel along W^2 in $\bar{M}^{n+\nu}$.*

Proof. It is sufficient to prove ii). In B_3, for $3 < r \leq n$, by (3.3) and (3.5) we have $\bar{D}e_r = -(\bar{c}/p)\omega_r e_3 + \sum \omega_{rt}e_t$. Thus, along W^2, $\bar{D}e_r = \sum_{t>3}^{n} \omega_{rt}e_t$, which shows that the tangent space in ii), i.e., the space spanned by $e_4, e_5, \cdots, e_n$, is parallel along W^2. q.e.d.

This theorem tells us how to construct a minimal submanifold in a space form as in the statement.

4. Minimal submanifolds with M-index 2, vanishing Q and ψ_v of rank 1

In this section, we shall investigate M^n in $\bar{M}^{n+\nu}$ as in Theorem 3 under the condition that ψ_v, $v \in \hat{N}$, $v \neq 0$, is of rank 1 everywhere. By this assumption and (1.13), we can choose frames b in B_3 such that

$$F = fe_{n+3}\,, \quad G = ge_{n+3}\,, \quad f^2 + g^2 \neq 0\,. \tag{4.1}$$

Denoting the set of these frames by B_4, from (1.12) we get

$$\begin{aligned} \lambda\omega_{n+1,n+3} &= f\omega_1 + g\omega_2\,, \qquad \mu\omega_{n+2,n+3} = g\omega_1 - f\omega_2\,, \\ \omega_{n+1,\gamma} &= \omega_{n+2,\gamma} = 0 \qquad (\gamma > n+3)\,. \end{aligned} \tag{4.2}$$

Theorem 4. *If M^n is minimal and of M-index 2 in $\bar{M}^{n+\nu}$ of constant curvature, ψ_v is of rank 1 for any nonzero $v \in \hat{N}$, and $Q \equiv 0$, then there exists a totally geodesic submanifold $\bar{M}^{n+3}$ of $\bar{M}^{n+\nu}$ containing M^n, in which M^n has the same properties*[4].

Proof. Using the same notations as in § 3, it is sufficient to show $\omega_{n+3,\gamma} = 0$ $(\gamma > n+3)$ in B_4. From (4.2), we get

$$\begin{aligned} d\omega_{n+1,\gamma} &= (1/\lambda)(f\omega_1 + g\omega_2) \wedge \omega_{n+3,\gamma} = 0\,, \\ d\omega_{n+2,\gamma} &= (1/\mu)(g\omega_1 - f\omega_2) \wedge \omega_{n+3,\gamma} = 0\,, \end{aligned}$$

which imply $\omega_{n+3,\gamma} = 0$ since $(f\omega_1 + g\omega_2) \wedge (g\omega_1 - f\omega_2) \neq 0$. q.e.d.

By virtue of the above theorem, we may put $\nu = 3$ in our case from the local point of view.

Lemma 11. *Under the conditions of Theorem 4, in B_4 we have the following:*

[4] We have supposed $n \geq 3$, but Theorem 4 is also true for $n = 2$.

(4.3) $$\{(d \log \lambda - pdv) - i(2\omega_{12} - \sigma\hat{\omega})\} \wedge (\omega_1 + i\omega_2) = 0 ,$$

(4.4) $$d\omega_{12} = -(p^2 + \bar{c} - \lambda^2 - \mu^2)\omega_1 \wedge \omega_2 ,$$

(4.5) $$d\hat{\omega} = -(1/(\lambda\mu))(2\lambda^2\mu^2 - f^2 - g^2)\omega_1 \wedge \omega_2 ,$$

(4.6) $$\begin{aligned}&\{d \log (f - ig) - d \log \lambda - pdv - i\omega_{12}\} \wedge (\omega_1 + i\omega_2)\\ &-\frac{i}{f - ig}\hat{\omega} \wedge \left\{f\left(\left(2\sigma - \frac{1}{\sigma}\right)\omega_1 + \frac{i}{\sigma}\omega_2\right)\right.\\ &\left.- ig\left(\frac{1}{\sigma}\omega_1 + i\left(2\sigma - \frac{1}{\sigma}\right)\omega_2\right)\right\} = 0 .\end{aligned}$$

Proof. By (3.3), (3.6) and $Q \equiv 0$, we get (4.3) immediately from (2.5). (4.4) and (4.5) are trivial from (2.8) and (2.9).

Now from (4.2) exterior derivation gives

$$\begin{aligned}&df \wedge \omega_1 + dg \wedge \omega_2 - (d \log \lambda + pdv) \wedge (f\omega_1 + g\omega_2)\\ &\quad - \left(\omega_{12} + \frac{1}{\sigma}\hat{\omega}\right) \wedge (g\omega_1 - f\omega_2) = 0 ,\\ &df \wedge \omega_2 - dg \wedge \omega_1 + (d \log \mu + pdv) \wedge (g\omega_1 - f\omega_2)\\ &\quad - (\omega_{12} + \sigma\hat{\omega}) \wedge (f\omega_1 + g\omega_2) = 0 ,\end{aligned}$$

which can be written as, in consequence of $d \log \mu = d \log \lambda + d \log \sigma$,

$$\begin{aligned}&\{d(f - ig) - (d \log \lambda + pdv + i\omega_{12})(f - ig)\} \wedge (\omega_1 + i\omega_2)\\ &\quad + \left(id \log \sigma - \frac{1}{\sigma}\hat{\omega}\right) \wedge (g\omega_1 - f\omega_2) - i\sigma\hat{\omega} \wedge (f\omega_1 + g\omega_2) = 0 .\end{aligned}$$

Since we have, from (2.7),

$$d \log \sigma \wedge \omega_1 = \left(\frac{1}{\sigma} - \sigma\right)\hat{\omega} \wedge \omega_2 , \qquad d \log \sigma \wedge \omega_2 = -\left(\frac{1}{\sigma} - \sigma\right)\hat{\omega} \wedge \omega_1 ,$$

substituting these in the above last equation we get (4.6).

Remark. $\hat{N} = \bigcup_{x \in M} \hat{N}_x$ introduced in § 2 is considered as a vector bundle over M^n with 2-dimensional fibre and has a metric connection induced from $\bar{M}^{n+\nu}$. $\hat{\omega} = \omega_{n+1,n+2}$ is its connection form and $d\hat{\omega}$ is its curvature form. Therefore $\hat{\omega}$ is a geometrical quantity of M^n in $\bar{M}^{n+3}$, which may be called the minimal torsion form of M^n.

Lemma 12. *Under the condition of Theorem* 4 *and the additional conditions*:

(α) $\hat{\omega} \neq 0$, *and* $\sigma = \mu/\lambda$ *is constant on* W^2,

(β) *W^2 is of constant curvature,*
where W^2 is an integral surface of the distribution $\mathfrak{w}$, for W^2 we have the following:

$$\sigma = 1 \text{ or } -1 \text{ and } 2\lambda^2 = p^2 + \bar{c} , \tag{4.7}$$

$$W^2 \text{ is flat}, \tag{4.8}$$

and, by supposing $\sigma = 1$ and $\omega_{12} = d\theta$ on W^2,

$$\hat{\omega} = 2d\theta , \tag{4.9}$$

$$dx = R((e_1^* + ie_2^*)d\bar{z}) , \tag{4.10}$$

$$\bar{D}(e_1^* + ie_2^*) = e_3 p dz + (e_{n+1}^* + ie_{n+2}^*)\lambda d\bar{z} , \tag{4.11}$$

$$\bar{D}e_3 = -pR((e_1^* + ie_2^*)d\bar{z}) , \tag{4.12}$$

$$\bar{D}(e_{n+1}^* + ie_{n+2}^*) = -(e_1^* + ie_2^*)\lambda dz + e_{n+3}\sqrt{2}\,\lambda d\bar{z} , \tag{4.13}$$

$$\bar{D}e_{n+3} = -\sqrt{2}\,\lambda R(e_{n+1}^* + ie_{n+2}^*)dz) , \tag{4.14}$$

where z is an isothermal coordinate of W^2 such that

$$\omega_1 + i\omega_2 = \exp(-i\theta)dz , \tag{4.15}$$

$$e_1^* + ie_2^* = \exp(i\theta)(e_1 + ie_2) , \; e_{n+1}^* + ie_{n+2}^* = \exp(2i\theta)(e_{n+1} + ie_{n+2}) . \tag{4.16}$$

Proof. From (2.7) and (α), we get $1 - \sigma^2 = 0$, i.e., $\sigma = 1$ or -1, so that we may suppose $\sigma = 1$. By means of (β), on W^2 we put $d\omega_{12} = -c\omega_1 \wedge \omega_2$, where c is a constant. Then (4.4) implies $2\lambda^2 = p^2 + \bar{c} - c$, and λ is constant on W^2 by Lemma 8 and Theorem 3. Therefore (4.3) implies $\hat{\omega} = 2\omega_{12}$ on W^2, from which we have $f^2 + g^2 = 2\lambda^2(\lambda^2 - c)$ by (4.5), so that $f^2 + g^2$ is also constant on W^2. Putting $f - ig = \sqrt{2}\,\lambda\sqrt{\lambda^2 - c}\exp(-i\varphi)$, from (4.6) we get the relation $\omega_{12} + \hat{\omega} + d\varphi = 0$, i.e., $3\omega_{12} + d\varphi = 0$. Thus we have $d\omega_{12} = d\hat{\omega} = 0$ on W^2, from which follows $c = 0$.

Hence W^2 must be flat, and we may put

$$f + ig = \sqrt{2}\,\lambda^2 \exp(-3i\theta) , \qquad \varphi = -3\theta . \tag{4.17}$$

On the other hand, we have $d(\omega_1 + i\omega_2) = -i\omega_{12} \wedge (\omega_1 + i\omega_2) = id\theta \wedge (\omega_1 + i\omega_2)$, and therefore there exists a local isothermal coordinate z as (4.15). Using (4.15) and (4.17), (4.2) can be written as

$$\omega_{n+1,n+3} + i\omega_{n+2,n+3} = \sqrt{2}\,\lambda \exp(-2i\theta)d\bar{z} \qquad \text{on } W^2 . \tag{4.18}$$

Now, to derive the Frenet formulas of W^2, we first have

$$dx = e_1\omega_1 + e_2\omega_2 = R((e_1 + ie_2)(\omega_1 - i\omega_2)) = R((e_1^* + ie_2^*)d\bar{z}) .$$

By means of (3.3), (1.10), (4.15) and (4.16), we obtain

$$\begin{aligned} \bar{D}(e_1 + ie_2) = & -(e_1 + ie_2)id\theta + e_3p(\omega_1 + i\omega_2) \\ & + (e_{n+1} + ie_{n+2})\lambda(\omega_1 - i\omega_2) , \end{aligned}$$

which is equivalent to (4.11). Analogously,

$$\bar{D}e_3 = -e_1\omega_{13} - e_2\omega_{23} = -pR((e_1^* + ie_2^*)d\bar{z}) .$$

From the relations

$$\begin{aligned} \bar{D}e_{n+1} &= -\lambda(e_1\omega_1 - e_2\omega_2) + 2e_{n+2}d\theta + e_{n+3}\omega_{n+1,n+3} , \\ \bar{D}e_{n+2} &= -\lambda(e_1\omega_2 + e_2\omega_1) - 2e_{n+2}d\theta + e_{n+3}\omega_{n+2,n+3} , \end{aligned}$$

it follows that

$$\begin{aligned} \bar{D}(e_{n+1} + ie_{n+2}) = & -(e_1 + ie_2)\lambda(\omega_1 + i\omega_2) - 2(e_{n+1} + ie_{n+2})id\theta \\ & + e_{n+3}(\omega_{n+1,n+3} + i\omega_{n+2,n+3}) , \end{aligned}$$

which is equivalent to (4.13) by (4.15), (4.16) and (4.18). Finally,

$$\begin{aligned} \bar{D}e_{n+3} &= -R((e_{n+1} + ie_{n+2})(\omega_{n+1,n+3} - i\omega_{n+2,n+3}) \\ &= -R((e_{n+1}^* + ie_{n+2}^*)\sqrt{2}\,\lambda dz) . \end{aligned}$$

5. Examples of minimal submanifolds of M-index 2

In this section, we shall find, as in Theorem 4, minimal submanifolds in space forms, for which a W^2 satisfies the conditions (α) and (β) in Lemma 12, and we shall suppose $n \geq 3$.

Case 1. *$\bar{M}^{n+3}$ is the Euclidean space E^{n+3}*. By Lemmas 12 and 8 the Frenet formulas for W^2 are

$$(5.1) \qquad \begin{aligned} dx &= R((e_1^* + ie_2^*)d\bar{z}) , \\ d(e_1^* + ie_2^*) &= e_3pdz + (e_{n+1}^* + ie_{n+2}^*)\lambda d\bar{z} , \\ de_3 &= -pR((e_1^* + ie_2^*)d\bar{z}) , \\ d(e_{n+1}^* + ie_{n+2}^*) &= -(e_1^* + ie_2^*)\lambda dz + e_{n+3}(\sqrt{2}\,\lambda d\bar{z}) , \\ de_{n+3} &= -\sqrt{2}\,\lambda R((e_{n+1}^* + ie_{n+2}^*)dz) , \end{aligned}$$

where

$$(5.2) \qquad p = 1/(a - v) , \quad \lambda = p/\sqrt{2} , \quad v < a , 0 < a .$$

From (5.1) it follows that $x + e_3/p$ is a fixed point and so we may suppose that it is the origin O of $E^{n+3} = R^{n+3}$. Then we have

$$x = -e_3/p \,. \tag{5.3}$$

From (5.1) again it is easily seen that e_3, $e_1^* + ie_2^*$, $e_{n+1}^* + ie_{n+2}^*$, e_{n+3} are all solutions of the partial differential equation

$$\frac{\partial^2 X}{\partial z \partial \bar{z}} = -\lambda^2 X \,.$$

Noticing this fact, we shall give a solution of (5.1).

In C^3 we choose 3 fixed constant vectors A_1, A_2 and A_3 such that

$$\begin{aligned} &A_j \cdot A_j = 0 \,, \qquad A_j \cdot A_k = A_j \cdot \bar{A}_k = 0 \,, \\ &A_1 \cdot \bar{A}_1 + A_2 \cdot \bar{A}_2 + A_3 \cdot \bar{A}_3 = 1/2 \,, \\ &\qquad\qquad i, k = 1, 2, 3;\ j \neq k \,, \end{aligned} \tag{5.4}$$

and put

$$\begin{aligned} U = \sum_{j=1}^{3} \{ &A_j \exp \lambda(z \exp(i\alpha_j) - \bar{z} \exp(-i\alpha_j)) \\ &+ \bar{A}_j \exp \lambda(-z \exp(i\alpha_j) + \bar{z} \exp(-i\alpha_j))\} \,, \end{aligned} \tag{5.5}$$

where the bar denotes the complex conjugate. It is clear that $U = \bar{U}$ and $U \cdot U = 1$ by (5.4). Next putting $\partial U/\partial \bar{z} = -\lambda\xi/\sqrt{2}$, we have

$$\begin{aligned} \xi = \sqrt{2} \sum_j \exp(-i\alpha_j)\{ &A_j \exp \lambda(z \exp(i\alpha_j) - \bar{z} \exp(-i\alpha_j)) \\ &- \bar{A}_j \exp \lambda(-z \exp(i\alpha_j) + \bar{z} \exp(-i\alpha_j))\} \,. \end{aligned} \tag{5.6}$$

It is easily seen that $\xi \cdot \bar{\xi} = 2$, $U \cdot \xi = 0$ and

$$\xi \cdot \xi = -4 \sum_j A_j \cdot \bar{A}_j (\cos 2\alpha_j - i \sin 2\alpha_j) \,. \tag{5.7}$$

Putting $\partial \xi/\partial \bar{z} = \lambda \eta$, we obtain

$$\begin{aligned} \eta = -\sqrt{2} \sum_j \exp(-2i\alpha_j)\{ &A_j \exp \lambda(z \exp(i\alpha_j) - \bar{z} \exp(-i\alpha_j)) \\ &+ \bar{A}_j \exp \lambda(-z \exp(i\alpha_j) + \bar{z} \exp(-i\alpha_j))\} \,, \end{aligned} \tag{5.8}$$

and therefore $\eta \cdot \bar{\eta} = 2$, $\xi \cdot \eta = \xi \cdot \bar{\eta} = 0$,

$$\eta \cdot \eta = 4 \sum_j A_j \cdot \bar{A}_j (\cos 4\alpha_j - i \sin 4\alpha_j) \,, \tag{5.9}$$

$$U \cdot \eta = \xi \cdot \xi / \sqrt{2} \,. \tag{5.10}$$

Finally putting $\partial \eta / \partial \bar{z} = \sqrt{2}\, \lambda V$, we have

$$V = \sum_j \exp(-3i\alpha_j)\{A_j \exp \lambda(z \exp(i\alpha_j) - \bar{z} \exp(-i\alpha_j)) \\ - \bar{A}_j \exp \lambda(-z \exp(i\alpha_j) + \bar{z} \exp(-i\alpha_j))\} . \tag{5.11}$$

Thus $V \cdot \bar{V} = 1$, $U \cdot V = U \cdot \bar{V} = 0$, $\eta \cdot V = \bar{\eta} \cdot V = 0$ and

$$V \cdot V = -2 \sum_j A_j \cdot \bar{A}_j(\cos 6\alpha_j - i \sin 6\alpha_j) , \tag{5.12}$$

$$\xi \cdot V = -\eta \cdot \eta / \sqrt{2} , \qquad \bar{\xi} \cdot V = -\xi \cdot \xi / \sqrt{2} . \tag{5.13}$$

By means of the above calculation, in addition to (5.4), if A_j, α_j, $j = 1, 2, 3$, satisfy

$$\sum_j A_j \cdot \bar{A}_j(\cos 2\alpha_j - i \sin 2\alpha_j) = 0 , \tag{5.14}$$

$$\sum_j A_j \cdot \bar{A}_j(\cos 4\alpha_j - i \sin 4\alpha_j) = 0 , \tag{5.15}$$

$$3\alpha_j \equiv \pi/2 \qquad (\mathrm{mod}\ \pi) , \tag{5.16}$$

then we obtain a solution of (5.1) by putting $e_3 = U$, $e_1^* + ie_2^* = \xi$, $e_{n+1}^* + ie_{n+2}^* = \eta$, $e_{n+3} = V$ and considering $C^3 = R^6$.

Condition (5.14) means that the broken segment $P_0P_1P_2P_3$ in the plane such that $P_{j-1}P_j = A_j \cdot \bar{A}_j$ and $\arg P_{j-1}P_j = 2\alpha_j$, $j = 1, 2, 3$, is closed, i.e., $P_0 = P_3$. Condition (5.15) also has an analogous meaning. By an elementary consideration, we see that the triangle $P_1P_2P_3$ must be equilateral, i.e.,

$$A_j \cdot \bar{A}_j = 1/6 , \qquad j = 1, 2, 3 . \tag{5.17}$$

Conversely, the above meanings are also sufficient for the validity of (5.14) and (5.15) respectively. Now, using the triangle $P_1P_2P_3$, and interchanging A_j with $\bar{A}_j$, $j = 1, 2, 3$, and the order of the index j, we may have the unique values of α_j, namely,

$$\alpha_1 = \pi/6 , \quad \alpha_2 = \pi/2 , \quad \alpha_3 = 5\pi/6 . \tag{5.18}$$

Thus we have a W^2 in $R^6 = C^3$ given by

$$x = -(a - v)\Big\{A_1 \exp \frac{i(u_1 + \sqrt{3}\, u_2)}{\sqrt{2}\,(a - v)} + \bar{A}_1 \exp \frac{-i(u_1 + \sqrt{3}\, u_2)}{\sqrt{2}\,(a - v)} \\ + A_2 \exp \frac{2iu_1}{\sqrt{2}\,(a - v)} + \bar{A}_2 \exp \frac{-2iu_1}{\sqrt{2}\,(a - v)} \\ + A_3 \exp \frac{i(u_1 - \sqrt{3}\, u_2)}{\sqrt{2}\,(a - v)} + \bar{A}_3 \exp \frac{-i(u_1 - \sqrt{3}\, u_2)}{\sqrt{2}\,(a - v)}\Big\} , \tag{5.19}$$

where $z = u_1 + iu_2$, and A_1, A_2, A_3 are complex vectors satisfying the conditions (5.4) and (5.17). Hence, by virtue of Theorem 3, we can construct a minimal submanifold M^n in E^{n+3}, as mentioned at the beginning of this section, as follows: Consider $E^{n+3} = R^{n+3} = R^6 \times R^{n-3}$, and take a W^2 given by (5.19) in R^6 and, at each point $y \in W^2$, the $(n-2)$-dimensional linear subspace $L^{n-2}(y)$ parallel to $e_3 = U$ and R^{n-3}. Then the locus of the moving $L^{n-2}(y)$ forms a submanifold M^n mentioned above.

Case 2. *$\bar{M}^{n+3}$ is the unit sphere S^{n+3}*. We may consider $S^{n+3} \subset E^{n+4}$. By putting $x = e_{n+4}$, the Frenet formulas for W^2 are

$$(5.20)\qquad \begin{aligned} dx &= R((e_1^* + ie_2^*)d\bar{z}) , \\ d(e_1^* + ie_2^*) &= e_3 p dz + (e_{n+1}^* + ie_{n+2}^*)\lambda d\bar{z} - e_{n+4}dz , \\ de_3 &= -pR((e_1^* + ie_2^*)d\bar{z}) , \\ d(e_{n+1}^* + ie_{n+2}^*) &= -(e_1^* + ie_2^*)\lambda dz + e_{n+3}(\sqrt{2}\,\lambda d\bar{z}) , \\ de_{n+3} &= -\sqrt{2}\,\lambda R(e_{n+1}^* + ie_{n+2}^*)dz) , \end{aligned}$$

where

$$(5.21)\qquad \begin{aligned} &p = \tan(v + a) , \qquad \lambda = 1/(\sqrt{2}\cos(v + a)) , \\ &\qquad 0 < v + a < \pi/2 , \qquad 0 < a < \pi/2 . \end{aligned}$$

From (5.20) it follows that $x + (1/p)e_3 = x + e_3 \cot(v + a)$ is a fixed point, so that $e_3 \cos(v + a) + e_{n+4} \sin(v + a) = e_0$ is a fixed unit vector and x is in an $(n+3)$-dimensional linear subspace E_1^{N+3} through the point $O_1 = e_0 \sin(v + a)$ and perpendicular to e_0. Thus W^2 lies in the $(n + 2)$-dimensional sphere $S^{n+3} \cap E_1^{n+3} = S_1^{n+2}(\cos(v + a))$ of radius $\cos(v + a)$, and we get $\overrightarrow{O_1x} = -e_3^* \cos(v + a)$, where $e_3^* = e_3 \sin(v + a) - e_{n+4} \cos(v + a)$. Using e_3^* we can easily obtain

$$\begin{aligned} d(e_1^* + ie_2^*) &= e_3^* \sqrt{2}\,\lambda dz + (e_{n+1}^* + ie_{n+2}^*)\lambda d\bar{z} , \\ de_3^* &= -\sqrt{2}\,\lambda R((e_{n+1}^* + ie_{n+2}^*)e\bar{z}) , \end{aligned}$$

and therefore the same equations with respect to $e_1^* + ie_2^*$, e_3^*, $e_{n+1}^* + ie_{n+2}^*$, e_{n+3} as (5.1). Hence we can take a W^2 in E_1^{n+3}, which is a solution of (5.20), and, at each point $y \in W^2$, an $(n-2)$-dimensional linear subspace $L^{*n-2}(y)$ in E_1^{n+3} through y as described in the previous case. Next, we project these $L^{*n-2}(y)$ onto S^{n+3} from O and denote the images by $L^{n-2}(y)$. The locus of the moving $L^{n-2}(y)$ forms a minimal submanifold M^n in S^{n+3}, which satisfies the conditions in Theorem 4 and (α) and (β) in Lemma 12.

Case 3. *$\bar{M}^{n+3}$ is the hyperbolic $(n + 3)$-space H^{n+2} of curvature* -1. We use the Poincaré representation of H^{n+3} in the unit disk in R^{n+3} with the canonical coordinates $x_1, \cdots, x_{n+3}$. The Riemannian metric H^{n+3} is given by

$$(5.22) \qquad ds^2 = 4dx\cdot dx/(1 - x\cdot x)^2 ,$$

where "$\cdot$" denotes the Euclidean inner product. Since the components of the Riemannian metric are

$$g_{ij} = 4\delta_{ij}/h^2 , \quad g^{ij} = h^2\delta^{ij}/4 , \quad h = 1 - x\cdot x ,$$

the Christoffel symbols are $\Gamma^k_{ij} = 2(\delta^k_i x_j + \delta^k_j x_i - \delta_{ij}x_k)/h$. For any vector field $X = \sum_j X^j\partial/\partial x_j$, its covariant differential with respect to the Riemannian connection of H^{n+3} is given by

$$(5.23) \qquad DX = h[a(2X/h) + 4\{(x\cdot X)dx - x(X\cdot dx)\}/h^2]/2 .$$

For any two tangent vector fields X, Y, we have $\langle X, Y\rangle = 4X\cdot Y/h^2$, where "$\langle , \rangle$" denotes the inner product in H^{n+3}. Therefore, if $b = (x, e_1, \cdots, e_{n+3})$ is an orthonormal base in H^{n+3}, then $(x, 2e_1/h, \cdots, 2e_{n+3}/h)$ is the one in R^{n+3}.

Now we describe the Frenet formulas for W^2 in H^{n+3} by means of the Poincaré representation (5.22). By putting

$$(5.24) \qquad \begin{aligned} \xi &= 2(e_1^* + ie_2^*)/h , & U &= 2e_3/h , \\ \eta &= 2(e_{n+1}^* + ie_{n+2}^*)/h , & V &= 2e_{n+3}/h , \end{aligned}$$

(4.10), $\cdots$, (4.14) become

$$(5.25) \qquad \begin{aligned} dx &= h(\xi d\bar{z} + \bar{\xi}dz)/4 , \\ d\xi &= \{Up - (x\cdot\xi)\bar{\xi}/2 + x\}dz + \{\eta\lambda - (x\cdot\xi)\xi/2\}d\bar{z} , \\ dU &= -\{p + (x\cdot U)\}(\xi d\bar{z} + \bar{\xi}dz)/2 , \\ d\eta &= -\{\xi\lambda + (x\cdot\eta)\bar{\xi}/2\}dz + \{V\sqrt{2}\,\lambda - (x\cdot\eta)\xi/2\}d\bar{z} , \\ dV &= -\{\eta\lambda/\sqrt{2} + (x\cdot V)\bar{\xi}/2\}dz - \{\bar{\eta}\lambda/\sqrt{2} + (x\cdot V)\xi/2\}d\bar{z} , \end{aligned}$$

in consequence of (5.23) and

$$\xi\cdot dx = h\{(\xi\cdot\xi)d\bar{z} + (\xi\cdot\bar{\xi})dz\}/4 = hdz/2 ,$$

where

$$(5.26) \qquad p = \coth(a - v) , \quad \lambda = \sqrt{p^2 - 1}/\sqrt{2} , \quad v < a .$$

On the other hand, any geodesic starting from the origin $O = (0, \cdots, 0)$ in H^{n+3} is a Euclidean straight line segment in the unit disk. The arc lengths v and r in H^{n+3} and R^{n+3} have the relation as $v = \log(1 + r)/(1 - r)$ and $r = \tanh(v/2)$. Since any W^2 is congruent to others under hyperbolic motions, we may suppose the focal point (z_0 in Theorem 3) of W^2 is the point O. Then we have

$$x = -Ur = -U \tanh(v/2) . \tag{5.27}$$

Replacing $a - v$ in (5.26) by v gives $h = 1 - x \cdot x = 1/\cosh^2(v/2)$, $2/h = \cosh v + 1, \lambda = 1/(\sqrt{2} \sinh v)$, $p - r = 1/\sinh v = \sqrt{2}\lambda$ and $x \cdot \xi = x \cdot \eta = x \cdot V = 0$, $x \cdot U = -r$ for W^2. Hence (5.25) is simplified as follows:

$$\begin{aligned} dx &= (\xi d\bar{z} + \bar{\xi} dz)/(2(1 + \cosh v)) , \\ d\xi &= U\sqrt{2}\lambda dz + \eta\lambda d\bar{z} , \\ dU &= -\sqrt{2}\lambda(\xi d\bar{z} + \bar{\xi} dz)/2 , \\ d\eta &= -\xi\lambda dz + V\sqrt{2}\lambda d\bar{z} , \\ dV &= -\sqrt{2}\lambda(\eta dz + \bar{\eta} d\bar{z})/2 . \end{aligned} \tag{5.28}$$

This system of equations except the first one is the system of equations (5.1) except its first one. Thus we see that we can construct a W^2 in H^{n+3} by making use of result in case $\bar{M}^{n+3} = E^{n+3}$. In fact, considering $R^{n+3} = R^6 \times R^{n-3}$, we take a surface W^2 satisfying (5.28), and, at each point y of W^2, the $(n - 2)$-dimensional linear subspace $\hat{L}^{n-2}(y)$ through y and parallel to U and R^{n-3}.

Let $L^{n-2}(y)$ be the totally geodesic subspace of H^{n+3} tangent to $\hat{L}^{n-2}(y)$ at y. Then the locus of the moving $L^{n-2}(y)$, $y \in W^2$, is a minimal submanifold M^n in H^{n+3}, which satisfies the required conditions.

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